THE KOSZUL COMPLEX IN PROJECTIVE DIMENSION ONE

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Let R be a noetherian ring and M a finite R-module. With a linear form χ on M one associates the Koszul complex $K(\chi)$. If M is a free module, then the homology of $K(\chi)$ is well-understood, and in particular it is grade sensitive with respect to $\operatorname{Im}\chi$.

In this note we investigate the case of a module M of projective dimension 1 (more precisely, M has a free resolution of length 1) for which the first non-vanishing Fitting ideal I_M has the maximally possible grade $r+1$, $r = \operatorname{rank} M$. Then $h = \operatorname{grade}\operatorname{Im}\chi \le r+1$ for all linear forms χ on M, and it turns out that $H_{r-i}(K(\chi)) = 0$ for all even $i < h$ and $H_{r-i}(K(\chi)) \cong \mathrm{S}^{(i-1)/2}(C)$ for all odd $i < h$ where S denotes symmetric power and $C = \operatorname{Ext}^1_R(M,R)$, in other words, $C = \operatorname{Cok}\psi^*$ for a presentation

$$0 \to F \xrightarrow{\psi} G \to M \to 0.$$

Moreover, if $h \le r$, then $H_{r-h}(K(\chi))$ is neither 0 nor isomorphic to a symmetric power of C, so that it is justified to say that $K(\chi)$ is grade sensitive for the modules M under consideration.

We furthermore show that the maximally possible value $\operatorname{grade}\operatorname{Im}\chi = r+1$ can only occur in two extreme cases: (i) $r = 1$ or (ii) $\operatorname{rank} F = 1$ and r is odd.

The note was motivated by a result of Migliore, Nagel, and Peterson (see [MNP], Proposition 5.1). They implicitly prove the result on $K(\chi)$ for Gorenstein rings R, using local cohomology. Our method allows more general assumptions. (Even the assumption that R is noetherian is superfluous if one uses the correct notion of grade.) It is based on results in [BV1] and has a predecessor in [HM]. The case in which $\operatorname{rank} F = 1$ has been treated in [BV3].

The situation in which the Fitting ideal I_M of M has only grade r is also of interest. For example, it occurs for the Kähler differentials of complete intersections with isolated singularities. While our method also yields results in this case, we have restricted ourselves to the case of grade $r+1$ for the sake of clarity.

The detailed account of the linear algebra of M and its exterior powers has been given in [BV2], Section 2.

For technical reasons we start with a situation dual to the above one. So let F, G be finite free R-modules of rank m, n and $\psi : G \to F$ an R-homomorphism. Set $\widehat{G} = G \otimes \mathrm{S}(F)$ where $\mathrm{S}(F)$ denotes the symmetric algebra of F. Then we may consider ψ an $\mathrm{S}(F)$-linear form on $\widehat{G}$ and can define the Koszul antiderivation

$$\partial_\psi : \bigwedge \widehat{G} \longrightarrow \bigwedge \widehat{G}$$

with respect to ψ in the usual way, i.e.

$$\partial_\psi(x_1 \wedge \ldots \wedge x_i) = \sum_{j=1}^{i} (-1)^{j+1} \psi(x_j) x_1 \wedge \ldots \widehat{x}_j \ldots \wedge x_i$$

for $x_1 \ldots x_i \in \widehat{G}$. We use the term Koszul complex also for the complex

$$0 \to R \xrightarrow{\varphi} G \xrightarrow{\varphi(1)\wedge} \bigwedge^2 G \xrightarrow{\varphi(1)\wedge} \bigwedge^3 G \xrightarrow{\varphi(1)\wedge} \ldots$$

associated with a linear map $\varphi : R \to G$. Suppose that $\psi\varphi = 0$ and let

$$d_\varphi : \bigwedge \widehat{G} \longrightarrow \bigwedge \widehat{G}$$

be the differential of the Koszul complex associated with $\varphi \otimes \mathrm{S}(F)$, i. e.

$$d_\varphi(x) = (\varphi(1) \otimes 1) \wedge x$$

for $x \in \bigwedge \widehat{G}$. Since $\psi\varphi = 0$, obviously $\partial_\psi d_\varphi + d_\varphi \partial_\psi = 0$. Thus we obtain the *Koszul*

bicomplex $\mathcal{K}$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "\varphi"] \arrow[d] & G \arrow[r] \arrow[d, "\psi"'] & \cdots \bigwedge^{p-1} G \arrow[r, "d_\varphi"] \arrow[d] & \bigwedge^p G \arrow[r] \arrow[d, "\partial_\psi"'] & \bigwedge^{p+1} G & \cdots \\
 & 0 \arrow[r] & F \arrow[r] \arrow[d] & \cdots M^{p-1,1} \arrow[r] \arrow[d] & M^{p,1} \arrow[r] \arrow[d] & M^{p+1,1} \arrow[d] & \cdots \\
 & & 0 \arrow[r] & \cdots M^{p-1,2} \arrow[r] \arrow[d] & M^{p,2} \arrow[r] \arrow[d] & M^{p+1,2} \arrow[d] & \cdots \\
 & & & \vdots & \vdots & \vdots & \\
 & & & 0 \arrow[r] & M^{p,p} \arrow[r] \arrow[d] & M^{p+1,p} \arrow[d] & \cdots \\
 & & & & 0 \arrow[r] & M^{p+1,p+1} \arrow[d] & \cdots \\
 & & & & & 0 & \cdots
\end{tikzcd}
$$

where

$$M^{p,q} = \bigwedge^{p-q} G \otimes \mathrm{S}^q(F)$$

for all integers p, q, and S^q means qth symmetric power. The row homology of $\mathcal{K}$ at $M^{p,q}$ is denoted by $H_\varphi^{p,q}$, the column homology by $H_\psi^{p,q}$. Thus $H_\varphi^{p,0}$ is the pth homology module H^p of the Koszul complex associated to φ. Set $N^p = \operatorname{Ker} \partial_\varphi^{p,0}$. The canonical injections $N^p \to \bigwedge^p G$ yield a complex homomorphism

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\bar{\varphi}"] \arrow[d, equal] & N^1 \arrow[r] \arrow[d] & \cdots N^p \arrow[r, "d_{\bar{\varphi}}"] \arrow[d] & N^{p+1} \cdots \arrow[d] \\
0 \arrow[r] & R \arrow[r, "\varphi"] & G \arrow[r] & \cdots \bigwedge^p G \arrow[r, "d_\varphi"] & \bigwedge^{p+1} G \cdots
\end{tikzcd}
$$

where the maps $\bar{\varphi}$, $d_{\bar{\varphi}}$ are induced by φ, d_φ. The homology of the first row at N^p is denoted by $\bar{H}^p$. We are now ready to state the key proposition. Here as in the following * means $\operatorname{Hom}_R(\ \ , R)$. Moreover, I_Ψ denotes the ideal of m-minors of (a matrix representing) ψ.

PROPOSITION 1. *Set* $g = \operatorname{grade} \mathrm{I}_M$, $C = \operatorname{Cok} \psi$, *and let* $h = \operatorname{grade} \operatorname{Im} \varphi^*$. *Assume that* $r = n - m \geq 1$ *and* $g = r + 1$. *Then*

(a) $\operatorname{Im} \varphi^* \subset \mathrm{I}_M$ *and, in particular,* $h \leq g$;

(b) $\bar{H}^i = 0$ *for* $0 \leq i \leq \min(2, h-1)$;

(c)

$$\bar H^i = \begin{cases} \mathrm{S}^{\frac{i-1}{2}}(C) & \text{if} \quad 3 \le i < h,\ i \not\equiv 0\ (2) \\ 0 & \text{if} \quad 3 \le i < h,\ i \equiv 0\ (2); \end{cases}$$

(d) *moreover, for $h \ge 3$ there is an exact sequence*

$$\begin{aligned} &0 \to \mathrm{S}^{\frac{h-1}{2}}(C) \to \bar H^h \to H^h \quad && \textit{if } h \not\equiv 0\ (2),\\ &0 \to \bar H^h \to H^h && \textit{if } h \equiv 0\ (2). \end{aligned}$$

Proof. Let $M = \operatorname{Cok}\psi^*$. We choose a basis $e_1, \dots, e_m$ of F^* and define the linear map $\Psi : G^* \to \bigwedge^{m+1} G^*$ by $\Psi(x) = \psi^*(e_1) \wedge \ldots \wedge \psi^*(e_m) \wedge x$. Then one obtains a complex

$$0 \to F^* \xrightarrow{\psi^*} G^* \xrightarrow{\Psi} \textstyle\bigwedge^{m+1} G^*$$

whose dual is the head of the Buchsbaum–Rim complex resolving $C = \operatorname{Cok}\psi$. It follows that $M^* = \operatorname{Im}\Psi^*$, and obviously $\operatorname{Im}\Psi^* \subset \mathrm{I}_M\, G$. Since $\varphi \in M^*$, one has $\operatorname{Im}\varphi \subset \mathrm{I}_M$. This shows (a).

We quote some well-known facts about the homology of $\mathcal{K}$. Let $0 \le p \le g$. Then $H^{p,q}_\psi = 0$ for $q \ne 0, p$ and $H^{p,p}_\psi = \mathrm{S}^p(C)$. (See [BV1], Proposition 2.1 for the general statement.) Furthermore $H^{p,q}_\varphi = 0$ for $p - q < h$ by the grade sensitivity of the Koszul complex for φ.

Claim (b) on $\bar H^i$ for $0 \le i \le \min(2, h-1)$ is easily proved from the long exact (co)homology sequence.

For (c) we modify the complex $\mathcal{K}$ to the complex $\widetilde{\mathcal{K}}$ by setting (i) $M^{p,p+1} = \mathrm{S}^p(C)$ and (ii) $M^{p,-1} = N^p$. The maps to be added are the natural surjection $M^{p,p} \to \mathrm{S}^p(C)$, the zero map $\mathrm{S}^p(C) \to M^{p+1,p+1}$, and those induced by the canonical injections $N^p \to \bigwedge^p G$. The truncation of $\widetilde{\mathcal{K}}$ to the "rectangle" $0 \le p \le h$, $0 \le q \le g$ has exact columns. Moreover, row homology for indices $< h$ can only occur at $M^{p,-1}$, namely $\bar H^p$, $0 \le p \le h$, and at $M^{p,p+1}$, namely $\mathrm{S}^p(C)$.

For an inductive argument we let $\mathcal{R}_q$, $q \ge -1$, be the qth row of $\widetilde{\mathcal{K}}$ and $\mathcal{B}_{q+1}$ be the image complex of $\mathcal{R}_q$ in $\mathcal{R}_{q+1}$. Then we have a series of exact sequences

$$0 \to \mathcal{B}_q \to \mathcal{R}_q \to \mathcal{B}_{q+1} \to 0.$$

Thus we can use the long exact (co)homology sequence for each q. With $E^{p,q} = H^p(\mathcal{B}_q)$ one therefore obtains the "southwest" isomorphisms

$$\bar H^i = E^{i,0} \cong E^{i-1,1} \cong \cdots \cong E^{\frac{i}{2},\frac{i}{2}}$$

if i is even, and

$$\bar H^i = E^{i,0} \cong E^{i-1,1} \cong \cdots \cong E^{\frac{i+1}{2},\frac{i-1}{2}}$$

if i is odd, $0 \le i < h$. In fact, there is an exact sequence

$$H^{i-(j+1),j} \to E^{i-(j+1),j+1} \to E^{i-j,j} \to H^{i-j,j}$$

and the extreme terms in this sequence are 0 for all j under consideration.

Let now i be even, and $j = i/2$. Since the map $M^{j,j} \to M^{j+1,j}$ is injective, the same holds true for its restriction $\mathcal{B}^j_j \to \mathcal{B}^{j+1}_j$ in $\mathcal{B}_j$, and so $\bar H^p = E^{j,j} = 0$.

In the case in which i is odd we can go one further step southwest, and obtain the isomorphism $\mathrm{S}^j(C) = E^{j,j+1} \cong E^{j+1,j}$ for $j = (i-1)/2$.

Since $H^h \neq 0$, we only have an exact sequence

$$0 \to E^{h-1,h} \to \bar{H}^h \to H^h,$$

but the arguments above can be applied to $E^{h-1,1}$; it is zero if h is even, and isomorphic to $\mathrm{S}^{(h-1)/2}(C)$ if h is odd.– □

REMARK 2. Proposition 1 shows that roughly the first half of the symmetric powers $\mathrm{S}^i(C)$, $i \le h$, can be interpreted as homologies of a Koszul complex. It is also possible to interpret the other half in a similar way. To this end we consider the column complexes $\mathcal{C}_0, \dots, \mathcal{C}_h$ of $\mathcal{K}$ (*not* of $\widetilde{\mathcal{K}}$) and set $\mathcal{C}_{h+1} = \mathrm{Cok}(\mathcal{C}_{h-1} \to \mathcal{C}_h)$. Then we obtain an exact sequence

$$0 \to \mathcal{C}_0 \to \cdots \to \mathcal{C}_h \to \mathcal{C}_{h+1} \to 0$$

of complexes, and the only nonzero (co)homology can occur along $\mathcal{C}_{h+1}$ and at $H^p(\mathcal{C}_p) \cong S_p(C)$ for $p > 0$. If one applies arguments similar to those in the proof of Proposition 1 (now proceeding in northwestern direction), then one obtains

$$H^i(\mathcal{C}_{h+1}) \cong \begin{cases} \mathrm{S}^{\frac{h+i}{2}}(C) & \text{if } h+i \equiv 0 \ (2), \\ 0 & \text{if } h+i \not\equiv 0 \ (2). \end{cases}$$

for $0 \le i \le h$. Note that the module $\mathcal{C}^i_{h+1}$ is resolved by $\mathcal{R}_i$, and $\mathcal{R}_i$ is just a truncated and shifted version of $\mathcal{R}_0 \otimes \mathrm{S}^i(F)$. The truncations of $\mathcal{R}_0$ resolve the exterior powers $\bigwedge^j M_\varphi$ where $M_\varphi = \mathrm{Cok}(\varphi)$. Thus $\mathcal{C}^i_{h+1} = \bigwedge^{h-i} M_\varphi \otimes \mathrm{S}^i(F)$.–

As in the proof of Proposition 1 let $M = \mathrm{Cok}\,\psi^*$. Then it is easy to see that $N^p = (\bigwedge^p M)^*$ for all p. In fact, ψ^* induces a presentation

$$\textstyle\bigwedge^{p-1} G^* \otimes F^* \to \bigwedge^p G^* \to \bigwedge^p M \to 0$$

for all p, and the exact sequence $0 \to N^p \to \bigwedge^p G \to \bigwedge^{p-1} G \otimes F$ is obtained by dualizing. It follows that N^r is free of rank 1 (provided $\operatorname{grade} \mathrm{I}_M \ge 2$), and $N^p = 0$ for $p > r$.

COROLLARY 3. *Let $\psi : G \to F$ be as above, and assume that $g = \operatorname{grade} \mathrm{I}_M = r+1$. Then the following conditions are equivalent.*

(1) There is a homomorphism $\varphi : R \to G$ such that $\psi\varphi = 0$ and the ideal $\mathrm{Im}\,\varphi^*$ *has grade $r+1$;*

(2) (i) $r = 1$ *or* (ii) $m = 1$ *and r is odd.*

Proof. The implication (2) ⇒ (1) is an easy exercise. (See also the considerations at the end of this note.)

For the converse observe that $N^p = 0$ for $p > r$ and that N^r is free of rank 1. So $\bar{H}^r$ must be cyclic. If r were even, then $\bar{H}^r = 0$ by Proposition 1, and $\mathrm{Im}\,\varphi^* = R$. Thus r must be odd. In this case $\bar{H}^r = \mathrm{S}^{(r-1)/2}(C)$ by Proposition 1 where $C = \mathrm{Cok}\,\psi^*$. So if $r \ge 3$, then C must be cyclic, which in turn means $m = 1$.– □

We now return to our original purpose. Therefore let $\bar\chi$ be a linear form on $M = \operatorname{Cok}\psi^*$. The induced linear form on G^* is denoted by χ; note that $\chi\psi^* = 0$. Set $\varphi = \chi^*$, and, as above, $r = n - m$. We want to connect the truncated Koszul complex

$$0 \to \bigwedge^r M \to \cdots \bigwedge^i M \xrightarrow{\partial_{\bar\chi}} \bigwedge^{i-1} M \cdots \to M \to R \to 0 \tag{1}$$

with the complex

$$0 \to R \xrightarrow{\bar\varphi} N^1 \to \cdots N^p \xrightarrow{d_{\bar\varphi}} N^{p+1} \cdots \tag{2}$$

considered above. We have already observed that $N^p = (\bigwedge^p M)^*$.

LEMMA 4. *With notation from above, let $g = \operatorname{grade} \mathrm{I}_M \geq r+1$. Then there are maps $\mu_p : \bigwedge^p M \to N^{r-p}$, $p = 0, \dots, r$, such that $d_{\bar\varphi}\mu_p = \pm\mu_{p-1}\partial_{\bar\chi}$ and which are isomorphisms for $p = 0, \dots, r-1$ and injective for $p = r$. If, in particular, $g = r+1 \geq 2$, then we obtain the following diagram of maps, the columns of which are exact and with commutative or anticommutative rectangles:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \bigwedge^r M \arrow[r] \arrow[d, "\mu_r"'] & \bigwedge^{r-1} M \arrow[r] \arrow[d, "\mu_{r-1}"'] & \bigwedge^{r-2} M \arrow[d] \\
0 \arrow[r] & R \arrow[r, "\bar\varphi"] \arrow[d] & N^1 \arrow[r] \arrow[d] & N^2 \\
0 \arrow[r] & R/\mathrm{I}_M \arrow[r] \arrow[d] & 0 & \\
 & 0 & &
\end{tikzcd}$$

Proof. As in [BV2], p. 17, we choose isomorphisms $\gamma : \bigwedge^m F^* \to R$, $\delta : \bigwedge^n G^* \to R$, and define maps

$$\nu_p : \bigwedge^p G^* \to (\bigwedge^{r-p} G^*)^*,$$

$p = 0, \dots, r$, by

$$\nu_p(x)(y) = \delta(x \wedge y \wedge \bigwedge^m \psi^*(z)),$$

$x \in \bigwedge^p G^*,\ y \in \bigwedge^{r-p} G^*,\ z = \gamma^{-1}(1)$. Via the natural isomorphism $(\bigwedge^{r-p} G^*)^* \cong \bigwedge^{r-p} G$ we regard ν_p as a map $\bigwedge^p G^* \to \bigwedge^{r-p} G$. One has $\operatorname{Im}\nu_p \subset N^{r-p}$, and it is an easy exercise to show that the diagram

$$\begin{tikzcd}
\bigwedge^p G^* \arrow[r, "\partial\chi"] \arrow[d, "\nu_p"'] & \bigwedge^{p-1} G^* \arrow[d, "\nu_{p-1}"'] \\
\bigwedge^{r-p} G \arrow[r, "d_{\chi^*}"] & \bigwedge^{r-p+1} G
\end{tikzcd}$$

is commutative or anticommutative (see for example [HM], proof of Theorem 3.1). Consequently the same is true for

$$\begin{array}{ccc} \bigwedge^p M & \xrightarrow{\partial\bar\chi} & \bigwedge^{p-1} M \\ {\scriptstyle\rho_p}\big\downarrow & & {\scriptstyle\rho_{p-1}}\big\downarrow \\ \operatorname{Im}\nu_p & \xrightarrow{d_{\chi^*}\,|\,\operatorname{Im}\nu_p} & \operatorname{Im}\nu_{p-1} \end{array}$$

where ρ_p and ρ_{p-1} are induced by ν_p and ν_{p-1}. Now let μ_p be the composition of ρ_p and the canonical injection $\operatorname{Im}\nu_p \to N^{r-p}$. Then the equation asserted in the proposition obviously holds. In case $p < \operatorname{grade} \mathrm{I}_M - 1$, $\operatorname{Im}\nu_p = N^{r-p}$. This proves the remaining statements.– $\square$

If we look at the homology of the truncated Koszul complex (1) associated to $\bar\chi$ instead of the homology of (2), then a somewhat smoother assertion than Proposition 1 is possible.

THEOREM 5. *Let M be module with a finite free resolution of length* 1, $M = \operatorname{Cok}\psi^*$ *where $\psi : G \to F$ is as above, and assume that* $g = \operatorname{grade} \mathrm{I}_M = r+1$. *Let $\bar\chi$ be a linear form on M. Then* $\operatorname{Im}\bar\chi \subset \mathrm{I}_M$, *and for the homology $\bar H_p$ at $\bigwedge M^p$ of the truncated Koszul complex (1) associated to $\bar\chi$ the following holds:*

$$\bar H_{r-i} = \begin{cases} \mathrm{S}^{\frac{i-1}{2}}(C) & \text{if } 0 \le i < h,\ i \not\equiv 0\ (2) \\ 0 & \text{if } 0 \le i < h,\ i \equiv 0\ (2), \end{cases}$$

where $\mathrm{S}^0(C) = R/\,\mathrm{I}_M$.

Furthermore there is a $\bar\chi$ with $\operatorname{grade}\operatorname{Im}\bar\chi = g$ *if and only if* (i) $r = 1$ *or* (ii) $m = 1$ *and r is odd. In this case we have necessarily* $\operatorname{Im}\chi = \mathrm{I}_M$.

Proof. Consider the diagram in Lemma 4. Since μ_r is injective, $\bar H_r$ must be zero if $h > 0$. Next let $h > 1$. Then we obtain $R/\,\mathrm{I}_M \cong \bar H_{r-1}$, since $\bar H^0 = \bar H^1 = 0$. Finally if $h > 2$, then in addition $\bar H^2 = 0$, so $\bar H_{r-2} = 0$ as stated. The remaining assertions concerning the homology of (1) are contained in Proposition 1.

Instead of $\bar\chi$ we can consider the induced linear form χ on G^*. Corollary 3 yields the statement about the existence of such a linear form χ satisfying $\operatorname{grade}\operatorname{Im}\chi = g$. Assume that such a χ exists. If $m = 1$, then $\operatorname{Im}\chi = \mathrm{I}_M$ by Proposition 2 in [BV3]. If $r = 1$, then, under our assumptions, $M = \operatorname{Cok}\psi^*$ is an ideal in R which must be isomorphic to $\operatorname{Im}\chi$. Using the Hilbert-Burch Theorem, we have $\operatorname{Im}\chi = a\,\mathrm{I}_M$ with an element $a \in R$. Since $\operatorname{grade}\operatorname{Im}\chi = 2$, a must be a unit.– $\square$

REMARK 6. (a) It is a noteworthy fact that the homology $\bar H_{r-i}$ of the truncated Koszul complex (1) associated to $\bar\chi$ is independent of χ for $i < h$.

(b) The Koszul complex associated to a linear form χ on a free module of rank r is grade sensitive: its homology vanishes for $j > r - \operatorname{grade}\operatorname{Im}\chi$, and does not vanish at $r - \operatorname{grade}\operatorname{Im}\chi$.

In a sense, this is also true for the linear form $\bar\chi$ considered in Theorem 5: of course, “vanishes” must be replaced by “vanishes if i is even and is isomorphic

to $\mathrm{S}^{(i-1)/2}(C)$ if i is odd". Then Theorem 5 covers all $i = 0, \dots, h-1$, but the analogy also persists if $i = h < g$. In fact, let $\mathfrak{p}$ be a prime ideal of grade h such that $\operatorname{Im}\chi \subset \mathfrak{p}$. The module $M_{\mathfrak{p}}$ is free and $\mathrm{S}^j(C_{\mathfrak{p}}) = \mathrm{S}^j(C)_{\mathfrak{p}} = 0$ for all j. Therefore one can apply the grade sensitivity of the Koszul complex for a free module, and $\bar{H}_h$ can be neither 0 nor isomorphic to a non-vanishing symmetric power of C: otherwise its localization would vanish.

(c) That we have truncated the Koszul complex of $\bar{\chi}$ is inessential. In fact, $\bigwedge^r M$ is torsionfree of rank 1, and $\bigwedge^{r+1} M$ has rank 0. Hence $\operatorname{Hom}_R(\bigwedge^{r+1} M, \bigwedge^r M) = 0$, and the homology of the full and of the truncated Koszul complexes at $\bigwedge^r M$ coincide.

Let $\psi : G \to F$ be as above, $r = n - m$, and $g = \operatorname{grade} \mathrm{I}_M = r + 1$. In case $r > 1$, the existence of a linear form χ on G^* with $\operatorname{grade}\operatorname{Im}\chi = g$ can be described equivalently and independently of the last theorem.

PROPOSITION 7. *With the assumptions on ψ and χ from Theorem 5, assume in addition that $r > 1$. Then* $\operatorname{grade}\operatorname{Im}\chi = g$ *is possible if and only if there exists a submodule U of $M = \operatorname{Cok}\psi^*$ with the following properties:*

(1) $\operatorname{rank} U = r - 1$*;*

(2) U is reflexive, orientable, and $U_{\mathfrak{p}}$ is a free direct summand of $M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of R such that $\operatorname{grade}\mathfrak{p} \le r$.

Proof. Let $\bar{\chi}$ be a linear form on M such that $\operatorname{grade}\operatorname{Im}\bar{\chi} = r + 1$. Set $U = \operatorname{Ker}\bar{\chi}$. Then (1) and the last property in (2) are obvious. Since $\operatorname{Im}\bar{\chi}$ is torsionfree and M is reflexive, U must be reflexive. Because $\operatorname{Im}\bar{\chi}$ has grade ≥ 2, it is orientable. M being orientable, the orientability of U follows from Proposition (2.8) in [B].

Conversely let U be a submodule of M which satisfies (1) and (2). Then $I = M/U$ is torsionfree of rank 1 and therefore an ideal in R which is orientable since U and M are orientable. Consequently $\operatorname{grade} I \ge 2$. So for a prime $\mathfrak{p}$ in R which contains I, the localization $IR_{\mathfrak{p}}$ cannot be free. On the other hand $I = R$ is impossible since $g = r + 1$. In view of the last condition in (2), I must have grade $r + 1$.– ☐

¿From Theorem 5 we know that the hypothesis of Proposition 7 can only be satisfied with $m = 1$ and r odd. The submodule U of M in Proposition 7 has projective dimension $r - 1$. In fact, the ideal $\mathrm{I}_M = \operatorname{Im}\chi$ is generated by $r + 1$ elements and has grade $r+1$. Therefore it is perfect, i. e. $\operatorname{projdim} R/I = r+1$. This implies $\operatorname{projdim} U = r - 1$ via the exact sequence $0 \to U \to M \to I \to 0$. Dualizing this exact sequence, we obtain an exact sequence $0 \to R \to M^* \to U^* \to 0$. Since $M^* \cong \bigwedge^{r-1} M$ has projective dimension $r - 1$, it follows that U^* has projective dimension $r - 1$.

The dualization argument just used amounts to interchanging the roles of ψ^* and χ, so that U^* plays the same role for χ^* and ψ as U for ψ^* and χ. If we choose χ in a special way, then we can even achieve that U and U^* are isomorphic in a skewsymmetric way: for the isomorphism $\sigma : U \to U^*$ one has $\sigma^* = -\sigma$ upon the identification of U and U^{**} via the natural isomorphism.

As we mentioned at the beginning of the proof of Corollary 3, it is easy to see that there is a linear form χ on G^* such that $\psi^*(1) \in \operatorname{Ker}\chi$ and $\operatorname{grade}\operatorname{Im}\chi = r + 1 = n$:

fix a basis $z_1, \dots, z_n$ of G^* and let $\psi^*(1) = \sum_{i=1}^n x_i z_i$; the map $\chi : \sum_{i=1}^n a_i z_i \mapsto \sum_{i=1}^n (-1)^i a_i x_{n+1-i}$ is an appropriate linear form. Let $\bar{\chi}$ be the induced form on $M = \operatorname{Cok} \psi^*$. The submodule $U = \operatorname{Ker} \bar{\chi}$ has the properties (1) and (2) of Proposition 7 (see the first part of the proof). Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \chi \arrow[r] \arrow[d, "\rho_1"'] & G^* \arrow[r, "\chi"] \arrow[d, "\rho"'] & R \arrow[d, equal] \\
0 \arrow[r] & N \arrow[r] & G \arrow[r, "\psi"] & R,
\end{tikzcd}$$

with exact rows. The isomorphism ρ is defined by $\rho(z_i) = (-1)^i z_{n+1-i}^*$ where $z_1^*, \dots, z_n^*$ is the basis of G dual to $z_1, \dots, z_n$, and ρ_1 is induced by ρ. Since $\rho_1(\psi^*(1)) = -\bar{\chi}$, we obtain a second commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R^* \arrow[r, "(\psi^*)_1"] \arrow[d] & \operatorname{Ker} \chi \arrow[r] \arrow[d, "\rho_1"'] & U \arrow[r] \arrow[d, "\bar{\rho}_1"'] & 0 \\
0 \arrow[r] & R^* \arrow[r, "\bar{\chi}^*"] & N \arrow[r] & U^* \arrow[r] & 0
\end{tikzcd}$$

with exact rows. $(\psi^*)_1$ is induced by ψ^* and the first vertical arrow maps 1 to -1. As ρ_1 is an isomorphism, the same is true for $\bar{\rho}_1$, so U turns out to be self-dual. Moreover $\bar{\rho}_1$ is skew-symmetric, i.e. $(\bar{\rho}_1)^* = -\bar{\rho}_1$ since the same is true for ρ.

Suppose that $R = K[X_1, \dots, X_n]$ is the polynomial ring in n indeterminates over a field K. If we define $\psi : R^n \to R$ by $\psi(e_i) = X_i$, then the module U is associated with a rank $n-2$ vector bundle on $\mathbb{P}^{n-1}(K)$. Such bundles exist however also for odd n; see [V]. The module V assocíated with the the construction in [V] is self-dual only for $n = 4$.

REFERENCES

1. W. Bruns. *The Buchsbaum-Eisenbud structure theorems and alternating syzygies.* Commun. Algebra **15** (1987), 873–925.

2. W. Bruns and U. Vetter. *Length formulas for the Local Cohomology of Exterior Powers.* Math Z. **191** (1986), 145–158.

3. W. Bruns, U. Vetter. *Determinantal rings.* Lect. Notes Math. **1327**, Springer 1988.

4. W. Bruns and U. Vetter. *A Remark on Koszul Complexes.* Beitr. Algebra Geom. **39** (1998), 249–254.

5. J. Herzog and A. Martsinkovsky. *Glueing Cohen-Macaulay modules with applications to quasihomogeneous complete intersections with isolated singularities.* Comment. Math. Helv. **68** (1993), 365–384

6. J.C. Migliore, U. Nagel, and C. Peterson. *Buchsbaum-Rim sheaves and their multiple sections.* J. Algebra **219** (1999), 378–420.

7. U. Vetter. *Zu einem Satz von G. Trautmann über den Rang gewisser kohärenter analytischer Garben.* Arch. Math. **24** (1973), 158–161.